Norm-Residue Theorem

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For a field k & ℓ s.t. $1/\ell \in k$.

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$$K^M_*(k)/\ell \xrightarrow{\cong} H^*_{et}(k; \mu_\ell^{\otimes *})$$

Milnor K-theory

Let k be a field. Milnor defined a graded ring $\mathrm{K}_*^M(k)$, called the **Milnor K-theory** of k, as follows:

- $\mathrm{K}_r^M(k) = 0$ for $r < 0$;
- $\mathrm{K}_0^M(k) := \mathbb{Z}$;
- $\mathrm{K}_1^M(k) = k^\times$; $k - 0$
- For $r \geq 2$, we define $\mathrm{K}_r^M(k) = \frac{\otimes_{i=1}^r k^*}{I}$, where I is the subgroup generated by elements of the form $a_1 \otimes \cdots \otimes a_r$ where $a_i + a_j = 1$ for some $i \leq j$. The class $\{a_1 \otimes \cdots \otimes a_r\}$ is typically denoted as $\{a_1, \cdots, a_r\}$. → symbols

The Milnor K-theory can be described in total as the quotient of the tensor algebra $T^*(k^\times)$ by the two sided ideal I generated by elements of the form $\{a, 1-a\}$ for $a \in k - \{0, 1\}$.

$= [a \otimes 1-a]$

$\Lambda^*(k^*)$

$k^\times$ $\frac{1}{1\otimes b = [0\otimes b] = 0}$ to identity

There are some immediate relations we can deduce from the definitions:

- Because $0 = \{1, b\}$, we have $\{a, b\} = -\{a^{-1}, b\}$. $\Longleftarrow \{aa^{-1}, b\} = 0$
- Because $-a = \frac{1-a}{1-a^{-1}}$, we have

$$\{a, -a\} = \{a, \frac{1-a}{1-a^{-1}}\} = \{a, 1-a\} + \{a, \frac{1}{1-a^{-1}}\}$$
$$= 0 - \{a, 1-a^{-1}\} = \{a^{-1}, 1-a^{-1}\} = 0.$$

- We have

$$\begin{aligned} 0 = \{ab, -ab\} &= \{a, -a\} + \{a, b\} + \{b, \underset{-a = -1\cdot a}{\underbrace{-a}}\} + \{b, b\} \\ &= 0 + \{a, b\} + \{b, a\} + \{b, -1\} + \{b, b\} \\ &= \{a, b\} + \{b, a\} + \underbrace{\{b, -b\}}_{=0} \\ &= \{a, b\} + \{b, a\} \end{aligned}$$

In particular, the third relation implies the symbols in $K_*^M(k)$ are alternating: For any permutation π with sign $(-1)^\pi$ we have

$$\{x_{\pi(1)}, \cdots, x_{\pi(n)}\} = (-1)^\pi \{x_1, \cdots, x_n\}.$$

Let us see a basic example of Milnor K-theory:

Proposition 1

Let $k = \mathbb{F}_q$ be a finite field. We have

$$\mathrm{K}_r^M(k) = 0, r \geq 2.$$

$r \geq 2.$

Remember that unit group of a finite field is always cyclic, so any element in $\mathrm{K}_2^M(k)$ can be written as

$$\{x^m, x^n\} = mn\{x, x\}.$$

$-1 = 1$

so, we just need to show that $\{x, x\} = 0$. If q is even number, we have $\{x, x\} = \{x, -x\} = 0$. If q is an odd number, we have $2\{x, x\} = 0$. Hence, for any odd integer m, n, it's true that $\{x, x\} = mn\{x, x\} = \{x^m, x^n\}$. Since the odd powers of x are classified as non-squares, it suffices to find a non-square u such that $1 - u$ is also a nonsquare. Notice the map $u \to 1 - u$ is an injection on the set $\mathbb{F}_q - \{0, 1\}$. There is $\frac{q-1}{2}$ nonsquares and $\frac{q-3}{2}$ squares, so necessarily some nonsquare will go to a nonsquare.

odd

Étale Cohomology and Galois Cohomology of a field

Let $X = \mathrm{Spec}(k)$. We consider the small étale site $X_{ét}$.

Proposition 2

Let $\bar{k}$ be the separable closure of k. There is an equivalence of categories ① between abelian sheaves over $X_{ét}$ and the category of continuous (Every element has an open stabilizer) ② $G = \mathrm{Gal}(\bar{k}/k)$-modules.

Proof.

$X_{ét}$ ① → ②

Let F be an abelian sheaf over ~~Sch_k~~. Let I be the poset of finite Galois extension of k in $\bar{k}$. Then we can set $M = \mathrm{colim}_{k' \in I} F(k')$. It has a G-action induced by the Gal(k'/k)-action on $F(k')$. $k' \to k$ $F(k) \to F(k')$

② → ①

On the other hand, given a continuous G-module M, for any finite separable extension k' of k, we define $F(k') = M^{\mathrm{Gal}(\bar{k}/k')}$, this defines a product preserving presheaf over $X_{ét}$ by remembering every object in $X_{ét}$ is a finite coproduct of affine schemes represented by finite separable extensions of k. To check the sheaf condition, it's enough to check for any finite separable extension k''/k', the following sequence

$$0 \to F(k') \to F(k'') \to F(k'' \otimes_{k'} k'') \cong F\Big(\prod_{\mathrm{Gal}(k''/k')} k''\Big) = \prod_{\mathrm{Gal}(k''/k')} F(k'')$$

is exact.

Galois extension.

finite étale k-alg. is product of finite separable extensions of k.

Proof Continued.

By construction, we want to check

$$0 \to M^{\mathrm{Gal}(\overline{k}/k')} \to M^{\mathrm{Gal}(\overline{k}/k'')} \to \prod_{\mathrm{Gal}(k''/k')} M^{\mathrm{Gal}(\overline{k}/k'')}$$

is exact. The first one is injective since $\mathrm{Gal}(\overline{k}/k'')$ is a subgroup of $\mathrm{Gal}(\overline{k}/k')$. The second map is $m \mapsto \prod_{\sigma\in\mathrm{Gal}(k''/k')}(m-\sigma(m))$, so its kernel is exactly

$$\ker = (M^{\mathrm{Gal}(\overline{k}/k'')})^{\mathrm{Gal}(k''/k')} = M^{\mathrm{Gal}(\overline{k}/k')}.$$

To check it gives an equivalence of categories, we need to see there are natural isomorphisms (exercises)

$$(\operatorname*{colim}_{i\in I} F(i))^{\mathrm{Gal}(\overline{k}/k')} = F(k')$$

and an isomorphism of G-modules

$$\operatorname*{colim}_{i\in I} M^{\mathrm{Gal}(\overline{k}/i)} \cong M.$$

□

Under this equivalence of categories, the global section function $F \mapsto F(k)$ is corresponding to the functor $M \mapsto M^{\mathrm{Gal}(\bar{k}/k)}$. Hence, for an an étale sheaf F over Spec(k), there is an isomorphism of cohomology

$R\Gamma^*(X, F) = R(-)^{\mathrm{Gal}(\bar{k}/k)}(M)$

$$H^*_{ét}(X; F) \cong H^*(\mathrm{Gal}(\bar{k}/k); F(k))$$

coh. $\mathbb{G}_m$

Now, let us consider the sheaves that is related to the Norm-residue theorem. For start, there is the multiplicative group scheme $\mathbb{G}_m$ defined by sending X to $\Gamma(X, \mathcal{O}_X)^*$. $\mathbb{G}_m = \mathcal{O}_X^*$

Let $l \in \mathbb{N}$ be an integer such that it's not equal to the characteristic of the field. So that l is invertible on Spec(k). Then we can define a map of Étale sheaves $l \colon \mathbb{G}_m \to \mathbb{G}_m$ by $x \in \mathbb{G}_m(U) \mapsto x^l \in \mathbb{G}_m(U)$.

$l \in k^*$

Proposition 3

There is a short exact sequence of Étale sheaves

$$0 \to \mu_l \to \mathbb{G}_m \to \mathbb{G}_m \to 0,$$

where $\mu_n(U) = \{x \in \Gamma(X, \mathcal{O}_X)^* | x^l = 1\}$.

$x \mapsto x^l$

surjective.

Proof.

By construction, μ_l is the kernel of the map. It's enough to show this is a surjective map of sheaves. To see this, we need to show for every $s \in \mathbb{G}_m(U)$, there is an open covering $\{U_i \to U\}$ such that $s|_{U_i}$ is in the image $l\colon \mathbb{G}_m(U_i) \to \mathbb{G}_m(U_i)$. Suppose $U = \mathrm{Spec}(A)$, we set $V = \mathrm{Spec}(A[T]/(T^l - s))$. The map $V \to U$ is surjective because the corresponding map is faithfully flat. Because the derivative of $T^l - s$ is lT^{l-1} is a unit, the ring map $A \to A[T]/(T^l - s)$ is a standard étale map by definition, which implies $V \to U$ is an open covering. $s|_V$ is in the image by construction. If U is not affine, we can consider the relative spectrum

$A \to A[T]$

$T \mapsto s$

$U \in R$

$T^l - T = s \in R$

$l: G_m(V) \to G_m(U)$

$$\pi\colon V = \underline{\mathrm{Spec}}_U(\mathcal{O}_U(t)/(t^l - s)) \to U$$

and restricting to its affine open subset. $\square$

Notice this sequence is not exact if we replace étale by Zariski.

The Kummer sequence indicates that there is a long exact sequence of cohomology groups

$H^2_{ét}(X;\mu_l)$

$$0 \to H^0_{ét}(X;\mu_l) \to H^0_{ét}(X;\mathbb{G}_m) \xrightarrow{n\,=\,l} H^0_{ét}(X;\mathbb{G}_m) \to H^1_{ét}(X;\mu_l) \to H^1_{ét}(X;\mathbb{G}_m) \to \cdots$$

$k^\times \xrightarrow{l} k^\times \qquad k^\times/l \qquad = 0$

For the 0-th cohomology, we have

$$H^0_{ét}(X;\mathbb{G}_m) = k^\times.$$

Hence, we have

$$H^0_{ét}(X;\mu_l) = \mathrm{Ker}(l\colon k^\times \to k^\times)$$

$a^l = 1$.

For a field k containing an l-th root of unity, we see that

$$H^0_{ét}(X;\mu_l) \cong \mathbb{Z}/l.$$

Otherwise, we have $H^0_{ét}(X;\mu_l) \cong 0$. For the first cohomology, we have the Hilbert 90:

$$H^1_{ét}(X;\mathbb{G}_m) \cong H^1(\mathrm{Gal}(\bar{k}/k);k^\times) = 0.$$

This implies that

$$H^1_{ét}(X;\mu_l) \cong k^\times/l. \quad = K^M_1(k)/l.$$

Étale Cohomology and Galois cohomology of a field

On étale cohomology, one can imagine one can define an external cup product:

$$H^n_{et}(X; F) \otimes H^m_{et}(X; G) \to H^{m+n}_{et}(X; F \otimes G)$$

This gives a graded ring (for $* = 0$, $\mu_l^{\otimes 0} := \mathbb{Z}/l$): $\mathbb{Z}/\ell$.

$$H^*_{ét}(X; \mu_l^{\otimes *}) = \bigoplus_m H^m_{ét}(X; \mu_l^{\otimes m}).$$

$K^M_*(k)/\ell \to H^*_{et}(k; \mu_\ell^{\otimes *})$

$= (T(k^\times)/I)/\ell$

Proposition 4

For $[a], [1-a] \in k^\times/l \cong H^1_{ét}(X, \mu_l)$ where $a \neq 1, 0$, we have a relation

$$[a] \cup [1-a] = 0 \in H^2_{ét}(X; \mu_l^{\otimes 2}).$$

$f: \mathrm{Spec}(\bar{k}) \to \mathrm{Spec}(k)$

$f^*: Sh(k) \rightleftarrows Sh(\bar{k}) : f_*$

$F \to f_*f^*F \to F$

$H^*(E; f_*f^*F) \cong H^*(k; f^*F)$ Leray spectral sequence.

Proof.

Let $\alpha = \sqrt[l]{a}$ and consider $E = k(\alpha)$. Then the inclusion $i: k \to E$ induces two natural maps on the étale cohomology groups $\mathrm{res}_{E/k}: H^*(k; \mu_l^{\otimes *}) \to H^*(E; \mu_l^{\otimes *})$ and $\mathrm{cores}_{E/k}: H^*(E; \mu_l^{\otimes *}) \to H^*(k; \mu_l^{\otimes *})$ that are compatible with cup product in the following way:

finite Galois extension

$$\mathrm{cores}_{E/k}(x) \cup y = \mathrm{cores}_{E/k}(x \cup \mathrm{res}_{E/k}(y)).$$

$\mathrm{cores} \circ \mathrm{res} = [E:k]$. on $H^*(k; \mu_\ell^{\otimes *})$ □

Étale Cohomology and Galois cohomology of a field

Proof continued.

In particular, for $* = 1$, the corestriction map is induced by the norm map $E \to k$. We have

Nm.

$$\mathrm{Nm}_{E/k}(1-\alpha) = \prod_{\sigma \in \mathrm{Gal}(E/k)} (1-\sigma(\alpha)) \overset{(}{=} 1-a.$$

$1-x^l$

$(1-a_1)\cdots(1-a_l)$

This implies

$$[a] \cup [1-a] = [a] \cup \mathrm{cores}_{E/k}([1-\alpha]) = \mathrm{cores}_{E/k}(\mathrm{res}_{E/k}([a]) \cup [1-\alpha]). \quad = 0$$

Notice that $\mathrm{res}_{E/k}([a]) = [\alpha^l] = 0 \in H^1(E;\mu_l) \cong E^\times/l$. ◻

Since the Milnor K-theory is described as the tensor algebra of $k^\times$ quotienting the relation $\{a, 1-a\}$. We see there is a natural ring map $\mathrm{K}_*^M(k) \to H^*_{\acute{e}t}(k;\mu_l^{\otimes *})$. Because the étale cohomology groups with μ_l-coefficient is always l-torsion, we see that the above map natural factors through $\mathrm{K}_*^M(k)/l$, which we call as the norm-residue map:

$$\mathrm{K}^M(k)/l \to H^*_{\acute{e}t}(k;\mu_l^{\otimes *})$$

Norm-residue map.

Theorem 1

$l \nmid$ ch k

Let k be a field and l be a positive integer that is not equal to the field characteristic. Then the norm-residue map is an isomorphism for every field k.

First Reductions: Transfer Argument

Consider the category of algebraic field extensions over k. Let F be a covariant functor on this category taking values in $\mathbb{Z}/l$-modules, and we also assume F is contravariant for finite field extensions k'/k. Hence, for a finite field extension $k \to k'$, we have a composite of maps $F(k) \to F(k') \to F(k)$, we require this map is multiplication by $[k' : k]$ on $F(k)$. If $[k' : k]$ is prime to l then we see $F(k)$ injects as a summand of ~~$F(k') = 0$~~. Hence, $F(k') = 0$ will imply $F(k) = 0$.

Proposition 5

Both $k \mapsto \mathrm{K}^M_m(k)/l$ and $k \mapsto H^m_{\acute{e}t}(k; \mu_l^m)$ are functors satisfying the hypothesis above. In particular, so do the kernel and cokernel of the norm-residue maps.

Proof.

Consider a finite field extension k'/k. For the functor $H^m_{\acute{e}t}(-; \mu_l^m)$, we have seen it has the restriction and corestriction. Sheaf-theorically, they are induced by ($F = \mu_l$):

$$F \to f_* f^* F \to F$$

Writing out the definition, one can see this is exactly $[k' : k]\,\mathrm{id}_F$. For the functor $K^M_m(-)$, it is obviously a covariant functor. The transfer map is induced via $\mathrm{Nm}_{k'/k}$ on degree 1. $\square$

$k \to \bar{k}$ k'

Using this argument, we may assume k contains all lth-roots of unity, that k is a perfect field, and even that k has no field extensions of degree prime to l.

First Reductions: Characteristic 0

Proposition 6

To prove the norm-residue theorem, it's enough to show the norm-residue map for fields k such that $\mathrm{char}(k) = 0$.

char(k)>0

Proof sketch.

By the transfer argument, we can suppose k is a perfect field. Let K be the fraction field of its Witt vectors $\mathbb{W}(k)$, in which case $\mathbb{W}(k)$ is a discrete valuation ring. By [Wei13, III.7.3], , one can define the specialization maps sp in this case, that are compatible with the norm-residue maps in the following sense:

$$\begin{tikzcd} K^M_m(K)/l \arrow[r, "\cong"] \arrow[d, "\mathrm{sp}"] \arrow[dr, phantom, "0 \ \text{inj}"] & H^m_{ét}(K;\mu_l^{\otimes m}) \arrow[d, "\mathrm{sp}"] \\ K^M_m(k)/l \arrow[r, "\text{surj}"'] & H^m_{ét}(K;\mu_l^{\otimes m}) \end{tikzcd}$$

Furthermore, we also know sp is a split surjection which is compatible with the norm-residue map. Because $\mathrm{Char}(K) = 0$, we know the top arrow is an isomorphism, which also implies the lower arrow is also an isomorphism. □

Connections to Motivic cohomology

Now, we will explain how the norm-residue theorem is connected to the motivic cohomology, where we let $X = \mathrm{Spec}(k)$. Recall that from last talk, we know

$$H^{p,q}(X,\mathbb{Z}) \cong \mathrm{CH}^q(X, 2q-p);$$

From [NS90], we have

Theorem 2

Let k be a field. We have $\mathrm{CH}^q(X,p) = 0$ for $p < q$ and $\mathrm{CH}^q(X,q) = \mathrm{K}_q^M(k)$.

Consider the cofiber sequence of motive spectra $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/l$, it induces a long exact sequence of motivic cohomology groups:

$$\cdots H^{p-1,p}(X;\mathbb{Z}/l) \to \overset{K_p^M(k)}{H^{p,p}(X;\mathbb{Z})} \xrightarrow{\times l} \underset{0}{\underset{\|}{\overset{K_p^M(k)}{H^{p,p}(X;\mathbb{Z})}}} \to H^{p,p}(X;\mathbb{Z}/l) \to \overset{0}{\overset{\|}{H^{p+1,p}(X;\mathbb{Z})}} \to \cdots$$

Since $H^{p+1,p}(X;\mathbb{Z}) \cong \mathrm{CH}^p(X;p-1) = 0$ and $H^{p,p}(X;\mathbb{Z}) \cong \mathrm{CH}^p(X;p) \cong \mathrm{K}_p^M(k)$ by the above theorem, we see that

$$H^{p,p}(X;\mathbb{Z}/l) \cong \mathrm{K}_p^M(k)/l.$$

In fact, following the same argument, we can see that

$$H^{p,q}(X;\mathbb{Z}/l) = 0 \qquad \text{for } p > q.$$

Connections to Motivic cohomology

To connect this to étale cohomology. We need to remember the other interpretation of motivic cohomology. Let X be a smooth variety. Then there is a motivic complex $\mathbb{Z}(q)$, which is a complex of étale sheaves with transfers (so they are also sheaves in Zariski and Nisnevich topology). The motive cohomology $H^{p,q}(X, \mathbb{Z})$ can be recognized as the hypercohomology of $\mathbb{Z}(q)$ over X in the Zariski topology. (Remark)
Consider the complex $\mathbb{Z}/l(q) = \mathbb{Z}/l \otimes \mathbb{Z}(q)$, it is still a complex of étale sheaves, and in fact, we have by [MVW06, 10.2]

1

$$H^*_{ét}(X; \mathbb{Z}/l(q)) \cong H^*_{ét}(X; \mu_l^{\otimes q}).$$

Consider the adjunction

$$L_{ét}\colon \mathrm{Sh}_{\mathrm{zar}}(X) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Sh}_{ét}(X)\colon i$$

If F is an étale sheaf, we have a Lerray spectral sequence

$$E_2^{p,q} = H^p_{\mathrm{Zar}}(X; R^q iF) \Rightarrow H^{p+q}_{ét}(X; F),$$

where the inclusion of the zero-th line gives us a natural change of topology morphism. Hence, for the motivic complex $\mathbb{Z}/l(q)$, we have

$$H^*_{\mathrm{Zar}}(X; \mathbb{Z}/l(q)) \to H^*_{ét}(X; \mathbb{Z}/l(q))$$

Let $* = q$, since we know $H^q_{\mathrm{Zar}}(X; \mathbb{Z}/l(q)) \cong H^{q,q}(X; \mathbb{Z}/l) \cong \mathrm{K}^M_q(k)/l$, we see this change of topology morphism recovers the norm-residue map.

The Hilbert 90 condition

Now, we will give a road map of the proof of the norm-residue theorem. We will mainly follow Chapter 1 of [HW19].
Because étale and Zariski cohomology over Spec(k) commutes with filtered limits, for any abilen groups A that can be written as a direct limit of $\mathbb{Z}$, we have

$$H^*_{zar\,/\acute{e}t}(X; A(i)) = H^*_{zar\,/\acute{e}t}(X; A \otimes \mathbb{Z}(i)) \cong H^*_{zar\,/\acute{e}t}(X; \mathbb{Z}(i)) \otimes A.$$

$X = \mathrm{Spec}(k)$

Definition 7

Fix n and l. We say that $H90(n)$ holds if $H^{n+1}_{\acute{e}t}(k, \mathbb{Z}_{(l)}(n)) = 0$ for any field $1/l \in k$.

When $n = 0$, we have $H^1(k, \mathbb{Z}) = H^1(\mathrm{Gal}(\bar{k}/k), \mathbb{Z}) = \mathrm{Hom}_{\mathrm{cont.}}(\mathrm{Gal}(\bar{k}/k), \mathbb{Z}) = 0$.
Which implies $H90(0)$ holds for any l.
When $n = 1$, we need to observe that $\mathbb{Z}(1) \cong \mathbb{G}_m[1]$. Hence, we have $H^2(k, \mathbb{Z}_{(l)}(1)) = H^2(k, \mathbb{G}_m[1])_{(l)} \cong H^1(k, \mathbb{G}_m)_{(l)} = 0$ by the Hilbert Theorem 90, which justifies the name.

Lemma 3

$H^n_{\acute{e}t}(k, \mathbb{Z}(m)) \otimes \mathbb{Z}_{(l)}$

For all $n > m$, the étale cohomology $H^n_{\acute{e}t}(k, \mathbb{Z}(m))$ is a torsion group, so its l-torsion subgroup is $H^n_{\acute{e}t}(k, \mathbb{Z}_{(l)}(m))$. When $1/l \in k$, we have $H^{n+1}_{\acute{e}t}(k, \mathbb{Z}_{(l)}(m)) \cong H^n_{\acute{e}t}(k, \mathbb{Q}/\mathbb{Z}_{(l)}(m))$. For $n = m$ we have an exact sequence

$$K^M_n(k) \otimes \mathbb{Q}/\mathbb{Z}_{(l)} \to H^n_{\acute{e}t}(k; \mathbb{Q}/\mathbb{Z}_{(l)}(n)) \to H^{n+1}_{\acute{e}t}(k; \mathbb{Z}_{(l)}(n)) \to 0.$$

The Hilbert 90 condition

Proof.

By [MVW06, 14.23] and [MVW06, 3.6], we have $H^n_{ét}(k;\mathbb{Q}(m)) \cong H^n(k;\mathbb{Q}(m))$ for all n. and $n > m$, $H^n(k,\mathbb{Q}(m)) = 0$, this implies

$$H^n_{ét}(k,\mathbb{Z}(m)) \otimes \mathbb{Q} \cong H^n_{ét}(k,\mathbb{Q}(m)) = 0.$$

0 0

Hence, we know $H^n_{ét}(k,\mathbb{Z}(m))$ is a torsion group. To see the isomorphism as claimed, we consider the long exact sequence induced by $0 \to \mathbb{Z}_{(l)} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}_{(l)} \to 0$ as follows:

0 0

$$\cdots \to H^n_{ét}(k,\mathbb{Q}(m)) \to H^n_{ét}(k,\mathbb{Q}/\mathbb{Z}_{(l)}(m)) \to H^{n+1}_{ét}(k,\mathbb{Z}_{(l)}(m)) \to H^{n+1}_{ét}(k,\mathbb{Q}(m)) \to \cdots.$$

We see the isomorphism by observing the first and the last cohomology groups are zero. To get the exact sequence, we consider the following commutative diagram:

$s,i\ K^M_n(k) \otimes \mathbb{Q}/\mathbb{Z}_{(l)}(n)$

$$\begin{tikzcd}
H^n_{zar}(k;\mathbb{Z}_{(l)}(n)) \arrow[r] \arrow[d] & H^n_{zar}(k;\mathbb{Q}(n)) \arrow[r] \arrow[d, "\cong"] & H^n_{zar}(k;\mathbb{Q}/\mathbb{Z}_{(l)}(n)) \arrow[r] \arrow[d] & 0 \\
H^n_{ét}(k;\mathbb{Z}_{(l)}(n)) \arrow[r] & H^n_{ét}(k;\mathbb{Q}(n)) \arrow[r] & H^n_{ét}(k;\mathbb{Q}/\mathbb{Z}_{(l)}(n)) \arrow[r, two heads] & H^{n+1}_{ét}(k;\mathbb{Z}_{(l)}(n))
\end{tikzcd}$$

0

This almost gives u the exact sequence by noticing that $K^M_n(k)/l \otimes \mathbb{Q}/\mathbb{Z}_{(l)} \cong H^n_{zar}(k;\mathbb{Q}/\mathbb{Z}_{(l)}(n))$. The exactness in the middle follows from a easy diagram chase. □

The Hilbert 90 condition

Theorem 4

Fix n and l. The condition $H90(n)$ holds if and only if the norm-residue map $K^M_n(k)/l \to H^n_{ét}(k;\mu_l^{\otimes n})$ is an isomorphism for every field k with $1/l \in k$.
In fact, $H90(n)$ holds implies that for any smooth scheme X over k and for all $p \leq n$, the change of topology map $H^p_{zar}(X;\mathbb{Z}/l(n)) \to H^p_{ét}(X;\mathbb{Z}/l(n))$ is an isomorphism.

proof for the if part.

Recall that $K^M_n(k) \cong H^n_{zar}(k;\mathbb{Z}(n))$. We have a commutative diagram induced by the change of topology map as follows

$$\begin{tikzcd}
K^M_n(k) \arrow[r, "l"] \arrow[d] & K^M_n(k) \arrow[r, two heads] \arrow[d] & K^M_n(k)/l \arrow[r] \arrow[d, "\cong"] & 0 \\
H^n_{ét}(k;\mathbb{Z}(n)) \arrow[r, "l"] & H^n_{ét}(k;\mathbb{Z}(n)) \arrow[r] & H^n_{ét}(k;\mu_l^{\otimes n}) \arrow[r, "0"] & H^{n+1}_{ét}(k;\mathbb{Z}(n)) \arrow[r, "l"] & \cdots
\end{tikzcd}$$

ker(ℓ) = ℓ-torsion

By assumption, the third vertical map is an isomorphism so by the commutative diagram, we see $H^n_{ét}(k;\mathbb{Z}(n)) \to H^n_{ét}(k;\mu_l^{\otimes n})$ is surjective. By exactness, we the next map is the zero map and the l-torsion part of $H^{n+1}_{ét}(k;\mathbb{Z}_{(l)}(n))$ is 0. By the lemma above, this is saying exactly

$$H^{n+1}_{ét}(k;\mathbb{Z}_{(l)}(n)) = 0.$$

The quick proof

Now, we can present a quick proof of the Norm-residue theorem with listing another two theorems

Definition 8

We say a field k containing $1/l$ is l-special if k has no finite field extensions of degree prime to l. Recall we can always assume k satisfies this condition by transfer arguement.

Theorem 5 Chapter 3.

Suppose that $H90(n-1)$ holds. If k is an l-special field and $\mathrm{K}_n^M(k)/l = 0$, then $H^n_{ét}(k, \mu_l^{\otimes n}) = 0$, which also implies $H^{n+1}_{ét}(k, \mathbb{Z}_{(l)}(n)) = 0$.

Theorem 6 Deep theorem.

Suppose that $H90(n-1)$ holds. Then for every field k of characteristic 0 and every nonzero symbol $a = \{a_1, \cdots, a_n\}$ in $\mathrm{K}_n^M(k)/l$, there is a smooth projective variety X_a whose function field $K_a = k(X_a)$ satisfies

- *a vanishes in $\mathrm{K}_n^M(K_a)/l$;*
- *the map $H^{n+1}_{ét}(k, \mathbb{Z}_{(l)}(n)) \to H^{n+1}_{ét}(K_a, \mathbb{Z}_{(l)}(n))$ is an injection.*

Root $\Rightarrow$ $\forall a \in \mathrm{K}_n^M(k)/l$ $\exists X_a$ satisfying the first condition

Open conjection iff the second condition is always true (Root variety)

The quick proof

Proof of the Norm-residue theorem.

By our reductions, we can assume k is an l-special field and has characteristic 0. For each $a \in \mathrm{K}_n^M(k)/l$, by Theorem 6, there is a smooth projective variety X_a such that a vanishes in $\mathrm{K}_n^M(k(X_a))/l$ and $H^{n+1}_{\acute{e}t}(k, \mathbb{Z}_{(l)}(n))$ embeds into $H^{n+1}_{\acute{e}t}(k(X_a), \mathbb{Z}_{(l)}(n))$. By putting an well-order of elements in $\mathrm{K}_n^M(k)/l$ and using a transfinite induction, we can get a sequence of field $\{k_\lambda\}$ such that a_λ vanaishes in $\mathrm{K}_n^M(k_\lambda)/l$ and $H^{n+1}_{\acute{e}t}(k_\lambda, \mathbb{Z}_{(l)}(n))$ embeds into $H^{n+1}_{\acute{e}t}(k_{\lambda+1}, \mathbb{Z}_{(l)}(n))$. Setting $k' = \cup_\lambda k_\lambda$, we see that $\mathrm{K}_n^M(k)/l \to \mathrm{K}_n^M(k')/l$ is a zero map and $H^{n+1}_{\acute{e}t}(k, \mathbb{Z}_{(l)}(n))$ embeds into $H^{n+1}_{\acute{e}t}(k', \mathbb{Z}_{(l)}(n))$. (Notice here we're using $H^{n+1}_{\acute{e}t}(k', \mathbb{Z}_{(l)}(n)) \cong \mathrm{colim}_\lambda\, H^{n+1}_{\acute{e}t}(k_\lambda, \mathbb{Z}_{(l)}(n))$ by Theorem 59.51.3 from stacks project.) Then, we can choose an l-special algebraic extension k'' of k'. By transfer argument, we know that

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$$H^{n+1}_{\acute{e}t}(k, \mathbb{Z}_{(l)}(n)) \hookrightarrow H^{n+1}_{\acute{e}t}(k', \mathbb{Z}_{(l)}(n)) \to H^{n+1}_{\acute{e}t}(k'', \mathbb{Z}_{(l)}(n))$$

is an injection and

$$\mathrm{K}_n^M(k)/l \xrightarrow{0\text{-map}} \mathrm{K}_n^M(k')/l \to \mathrm{K}_n^M(k'')/l \quad = 0$$

is a zero map.

Let $k^1 = k''$, and we iterate this construction to obtain an ascending sequence of field extensions k^m. Let L be the union of all k^m. Then L is l-special and $\mathrm{K}_n^M(L)/l = 0$ by construction, so $H^{n+1}_{\acute{e}t}(L, \mathbb{Z}_{(l)}(n)) = 0$ by Theorem 5. Since $H^{n+1}_{\acute{e}t}(k, \mathbb{Z}_{(l)}(n))$ embeds into $H^{n+1}_{\acute{e}t}(L, \mathbb{Z}_{(l)}(n))$, we finish the proof by Theorem 4. $\square$

$H^{*,*}(k;\mathbb{Z}/n)$

Corollary 7

Let k be a field containing a primitive l-th root of unit, then there is a ring isomorphism

$$H^{*,*}(k,\mathbb{Z}/l) \cong \mathrm{K}_*^M(k)/l[\tau],$$

where $\tau \in H^{0,1}(k,\mathbb{Z}/l) \cong H^0(k,\mu_l) \cong \mathbb{Z}/l$ is the class representing a primitive l-th root of unity.

$\mu_\ell^{\otimes q} \cong \mathbb{Z}/\ell$

$\mu_\ell^{\otimes q} \cong \mu_\ell^{\otimes q}$

Proof.

By the norm-residue theorem and Theorem 4, we have learned that

$$H^{p,q}(k,\mathbb{Z}/l) \cong \begin{cases} H^p_{ét}(k,\mu_l^{\otimes q}) & p \geq q; \\ 0 & p < q \end{cases}$$

Under the equivalence between étale sheaves and Galois modules, we see μ_l is equivalent to the trivial $\mathrm{Gal}(\overline{k}/k)$-module $\mathbb{Z}/l$ because the l-th root of unity is in k. Hence, the multiplication by a primitive l-th root of unity induces an isomorphism of $\mathrm{Gal}(\overline{k}/k)$-modules $(\mathbb{Z}/n)^{\otimes p} \otimes \mathbb{Z}/n \cong (\mathbb{Z}/n)^{\otimes p+1}$. In sheaf cohomologies, this gives an isomorphism $\tau\colon H^*_{ét}(k;\mu_l^{\otimes q}) \to H^*_{ét}(k;\mu_l^{\otimes q+1})$.
Then the norm-residue theorem and the identification of motivic cohomology with étale cohomology finishes the proof immediately. □

Reference

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